Theorem: Prove that A field has no divisors of zero.

Proof:-

Let F be a field. Then every non-zero element of F possesses multiplicative inverse.

Let $a, b \in F$ with $a \neq 0$

Consider, $ab = 0$

$\Rightarrow a^{-1}(ab) = a^{-1}0$ $\{\because a \neq 0, a \in F \Rightarrow a^{-1}$ exists$\}$

$\Rightarrow (a^{-1}a)b = 0$

$\Rightarrow 1 \cdot b = 0$

$\Rightarrow b = 0$

Similarly, let $ab = 0$ with $b \neq 0$ ~~and~~

~~Consider~~ Since, $ab = 0$

$\Rightarrow (ab)b^{-1} = 0b^{-1}$ $\{b \neq 0 \Rightarrow b^{-1}$ exists$\}$

$\Rightarrow a(bb^{-1}) = 0$

$\Rightarrow a \cdot 1 = 0$

$\Rightarrow a = 0$

$\therefore$ we see that if $a, b \in F$ and $ab = 0$ then either $a = 0$ or $b = 0$

Hence, F has no zero divisors.

Theorem: Prove that every field is an integral domain but the Converse is not necessarily true

Proof:- Let a field F is a Commutative ring with unity.

$\therefore$ In order to Prove that every field is an integral domain,

we should show that a field has no zero divisor

(Same as above theorem)

Let $a, b \in F$ with $a \neq 0$ such that $ab = 0$
We shall show that $b = 0$.
Since, $a \neq 0$, a^{-1} exists and
we have, $ab = 0$
$\Rightarrow a^{-1}(ab) = a^{-1}0$
$\Rightarrow (a^{-1}a)b = 0$
$\Rightarrow 1.b = 0 \Rightarrow b = 0$.
Similarly let $b \neq 0$ and $ab = 0$
Then, $ab = 0$
$\Rightarrow (ab)b^{-1} = 0b^{-1}$ $\{b \neq 0 \Rightarrow b^{-1}$ exists$\}$
$\Rightarrow a(bb^{-1}) = 0$
$\Rightarrow a \cdot 1 = 0 \Rightarrow a = 0$.
If $a, b \in F$, then $ab = 0 \Rightarrow a = 0$ or $b = 0$
Hence, A field has no zero divisors.

Therefore, every field is an integral domain
But the converse is not true.
It is shown by an example.
The ring of integers I is a
Commutative ring with unity.
Also, I does not possesses zero divisors.
We know that ~~any~~ if
$a, b \in I$ such that $ab = 0$
then either a or b must be zero.
Hence, the ring of integers I is an
integral domain but it is not a field.
Since the multiplicative inverse of any
non-zero integer $\in I$ does not belong to I.

Theorem:- Prove that every finite integral domain is a field
OR A finite Commutative ring without zero divisors is a field.

Proof:- Let D be a finite commutative ring without zero divisors having n elements $a_1, a_2, a_3, \ldots, a_n$.
In order to Prove D is a field
(i) we must produce an element $1 \in D$ such that $1 \cdot a = a \ \forall a \in D$
(ii) we should show that every non-zero element of D has an inverse.

Let $a \neq 0 \in D$
Consider the products $aa_1, aa_2, aa_3, \ldots, aa_n$.
All these are elements of D is closed w.r.to multiplication. because D is an integral domain.
All these elements are distinct.

Suppose on Contrary that $aa_i = aa_j$ for $i \neq j$
$\Rightarrow a(a_i - a_j) = 0$
Since D is without zero divisors and $a \neq 0$
so, $a_i - a_j = 0 \Rightarrow a_i = a_j$ Contradicting $i \neq j$

Hence $aa_1, aa_2, aa_3, \ldots, aa_n$ are all the n distinct elements of D placed in some order.

So, one of these elements will be equal to a.

Thus there exists an element say a_p such that
$$aa_p = a = a_p a \quad \{\because D \text{ is Commutative}\} \quad \text{—(i)}$$

We shall show that a_p is multiplicative identity of D

Let $y \in D$.

Then, from above discussion for some $x \in D$ we have
$$ax = y = xa \quad \text{—(ii)}$$

Now,
$$a_p y = a_p(ax) \quad \{\because ax = y\}$$
$$= (a_p a)x$$
$$= ax \quad \{\because a_p a = a\}$$
$$= y = y a_p \quad \{\because D \text{ is Commutative}\}$$

Thus,
$$a_p y = y = y a_p \quad \forall\, y \in D$$

Therefore, a_p is the unity element of ring D.

Let us denote it by 1.

Now, $1 \in D$

Therefore from the above discussion, one of the n products $aa_1, aa_2, \ldots, aa_n$ will be equal to 1.

Thus there exists an element say $b \in D$ such that $ab = 1 = ba$

$\therefore$ b is the multiplicative inverse of the non-zero element $a \in D$

Thus, every non-zero element of D is inversible

Hence, D is a field.